SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2011
Trial Higher School Certificate

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 2 Hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- All necessary working should be shown in every question, if full marks are to be awarded.
- Marks may NOT be awarded for messy or badly arranged work.
- All answers to be given in simplified exact form unless otherwise stated.

Total Marks – 84

- Attempt all questions 1-7
- All questions are of equal value.
- Start each new question in a separate answer booklet.
- Hand in your answers in 7 separate bundles: Questions 1, 2, 3, 4, 5 , 6 and 7.
- The mark value of each question is given in the right margin.

Examiner: *R Boros*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

Start a new booklet.

Question 1 (12 marks). **Marks.**

a) Solve $x(3-2x) > 0$. 2

b) Find $\frac{d}{dx}\left(e^{-x}\cos^{-1}x\right)$. 2

c) The remainder when $x^3 + ax^2 - 3x + 5$ is divided by $(x+2)$ is 11. Find the value of a. 2

d) Using the table of standard integrals, find the exact value of: 2

$$\int_0^{\frac{\pi}{8}} \sec 2x \tan 2x \, dx.$$

e) Solve for x, $\frac{x^2-9}{x} \geq 0$. 2

f) Find $\int_0^2 \frac{1}{4+x^2}\,dx$, leaving your answer in exact form. 2

End of Question 1.

Start a new booklet.

Question 2 (12 marks). **Marks.**

a) Use the substitution $x = \ln u$ to find: 3

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx.$$

b) Use one application of Newton's method to find an approximation to the root of the equation $\cos x = x$ near $x = \frac{1}{2}$. Give your answer correct to 2 decimal places. 2

c) The curves $y = e^{2x}$ and $y = 1 + 4x - x^2$ intersect at the point $(0,1)$. Find the angle, to the nearest minute, between the 2 curves at this point of intersection. 3

d) Prove that $\dfrac{2}{\tan A + \cot A} = \sin 2A$. 2

e) Find the derivative of $\cos^3 x^\circ$. 2

End of Question 2.

Start a new booklet.

Question 3 (12 marks). **Marks.**

a) (i) Expand $\cos(\alpha+\beta)$ 1

(ii) Show that $\cos 2\alpha = 1 - 2\sin^2\alpha$ 1

(iii) Evaluate $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$ 1

b) If $\alpha = \tan^{-1}\left(\frac{5}{12}\right)$ and $\beta = \cos^{-1}\left(\frac{4}{5}\right)$, calculate the exact value of $\tan(\alpha-\beta)$. 2

c) $A(-1,7)$ and $B(5,-2)$ are 2 points. Point P divides AB in the ratio $k:1$.

(i) Write down the coordinates of P in terms of k. 2

(ii) If P lies on the line $5x-4y-1=0$, find the ratio of $AP:PB$. 1

d) Use mathematical induction to prove that: 3

$$1\times 1! + 2\times 2! + 3\times 3! + \ldots + n\times n! = (n+1)! - 1$$

where n is a positive integer.

e) State the domain of $y = 2\sin^{-1}(1-x)$. 1

End of Question 3.

Start a new booklet.

Question 4 (12 marks). **Marks.**

a) $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ are 2 points on the parabola $x^2 = 4ay$.

(i) Show that the equation of PQ is given by: $y - \frac{1}{2}(p+q)x + apq = 0$ 2

(ii) Find the condition that PQ passes through the point $\left(0, -a\right)$. 1

(iii) If the focus of the parabola is S and PQ passes through $\left(0, -a\right)$, 2

prove that $\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$.

b) A family consists of a father, mother, 3 girls and 4 boys.

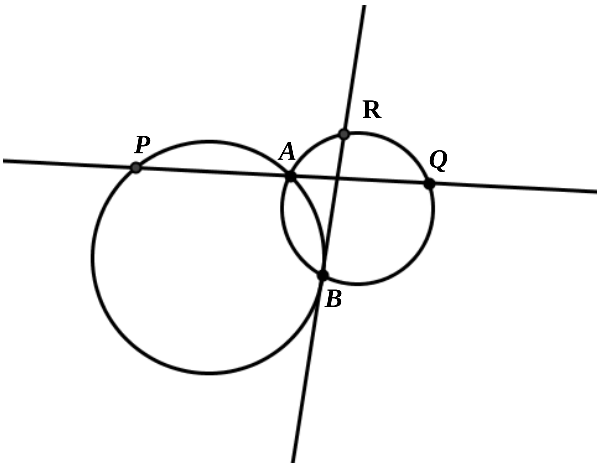
(i) If the family is seated at random along a bench, find the probability that the parents are at the ends, and the boys and girls are seated alternately between them. 1

(ii) If the same family is seated randomly at a round table, find the probability that the parents are separated by exactly two seats, and these seats are both occupied by boys. 1

Question 4 continued on next page.

Question 4 continued. **Marks.**

c) Two circles cut at A and B. A line through A meets one circle at P. Also, BR is a tangent to the circle ABP and R lies on the circle ABQ.



(i) Copy the diagram showing the above information.

(ii) Prove that $PB \| QR$. 2

d) One root of $x^3 + px^2 + qx + r = 0$ equals the sum of the two other roots. 3

Prove that $p^3 + 8r = 4pq$.

End of Question 4.

Start a new booklet.

Question 5 (12 marks). **Marks.**

a) The area bounded by the curve $y = \sin^{-1} x$, the y-axis and $y = \frac{\pi}{2}$ is rotated about the y-axis.

(i) Show that the volume of the solid so formed is given by: 1

$$\pi \int_0^{\frac{\pi}{2}} \sin^2 y \, dy.$$

(ii) Hence, find the exact volume of this solid. 2

b) The area of an equilateral triangle is increasing at the rate of 4 cm^2/s.

(i) If x is the length of the side of the triangle, find an expression for the area of the triangle. 1

(ii) Find the exact rate of increase of the side of the triangle, when it has a side length of 2 cm. 2

Question 5 continued on next page.

Question 5 continued. **Marks.**

c) (i) Find the largest possible domain of positive values for which $f\left(x\right)=x^2-6x+13$ has an inverse. 1

(ii) Find the equation of the inverse function $f^{-1}\left(x\right)$. 2

d) (i) Prove that $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)=\ddot{x}$. 1

(ii) A cat moving in a straight line has an acceleration given by $\ddot{x}=x(8-3x)$, where x is the displacement in metres from a fixed point O. Initially the cat is at the origin, O, and has a speed of 4 m/s. Find the cat's speed when it is 1 m on the positive side of O. 2

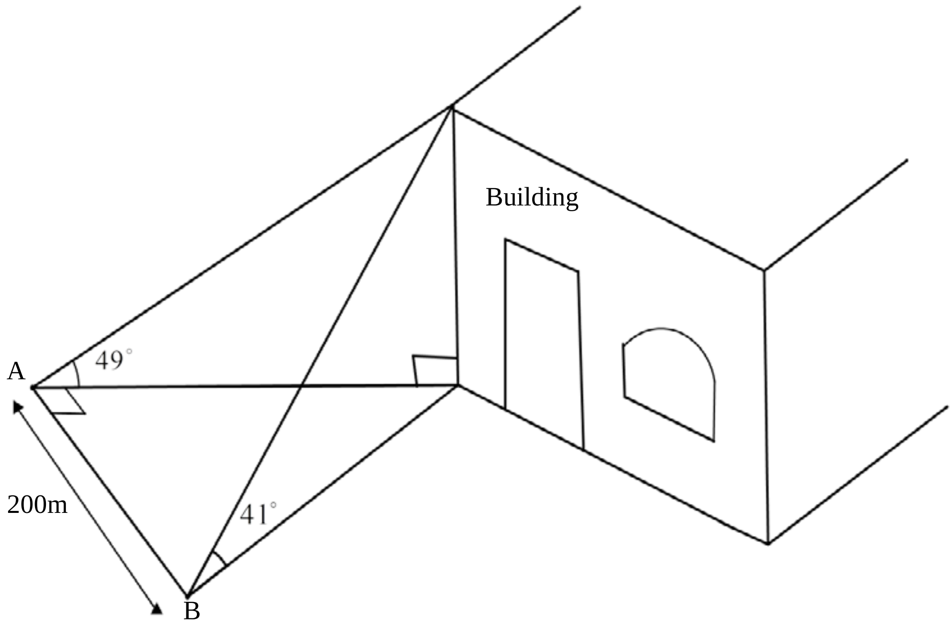
End of Question 5.

Start a new booklet.

Question 6 (12 marks). **Marks.**

a) Given $f(x) = 3\cos^{-1}(\sin 2x) - 2\sin^{-1}(\cos 3x)$, show that $f(x)$ is a constant function by finding $f'(x)$. 2

b) From point *A*, due south of a building, the angle of elevation to the top of the building is 49°. Two hundred metres due east of *A*, at point *B*, the angle of elevation is found to be 41° as shown in the diagram below. Determine, to the nearest metre, the height of the building. 3



Question 6 continued on next page.

Question 6 continued. **Marks.**

c) A particle is oscillating in simple harmonic motion such that its displacement x metres from a given origin O satisfies the equation $\frac{d^2x}{dt^2} = -4x$, where t is the time in seconds.

(i) Show that $x = a\cos(2t + \beta)$ is a possible equation of motion for the particle, where a and β are constants. 1

(ii) The particle is observed at time $t = 0$ to have a velocity of 2 m/s and a displacement from the origin of 4 m. Find the amplitude of the oscillation. 2

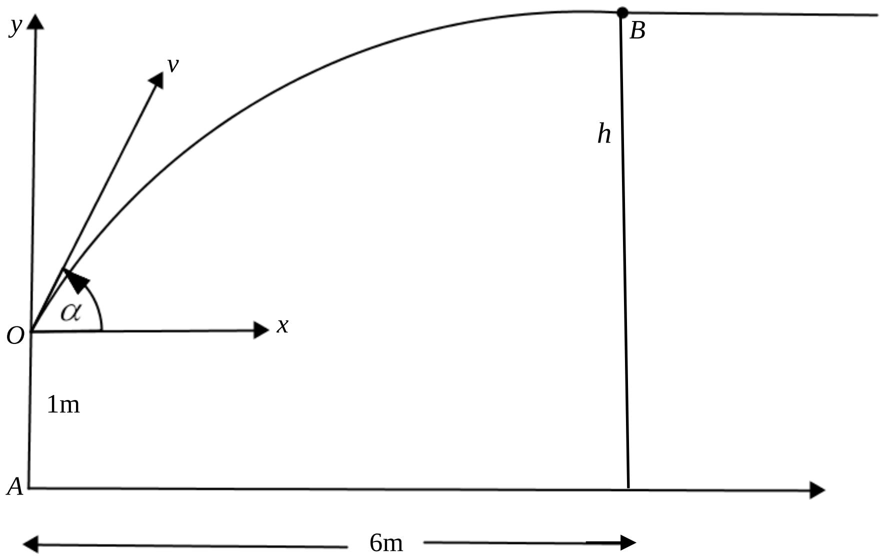
(iii) Determine the maximum velocity of the particle. 1

d) Using $t = \tan\frac{\theta}{2}$, find the general solution in radians to $\sin\theta - \cos\theta = 1$. 3

End of Question 6.

Start a new booklet.

Question 7 (12 marks). **Marks.**



a) A girl, 1 metre tall, throws a frisbee from a point *O*, with velocity *v* m/s at an angle α with the horizontal. It strikes the wall of a building at the highest point *B* of its trajectory. It takes $1\frac{1}{2}$ seconds to travel from *O* to *B* and the wall is 6 metres away from the girl. Taking the coordinate axes from *O* as shown and $g = 10$ m/s^2.

(i) Prove that at any time t , the position of the frisbee is given by: 2

$$x = vt\cos\alpha \text{ and } y = vt\sin\alpha - 5t^2 .$$

(ii) Show that $v\cos\alpha = 4$ and $v\sin\alpha = 15$. 2

(iii) Determine the initial velocity *v*, in exact form, and the angle of projection α to the nearest degree. 2

(iv) Find the height *h* of the building, correct to 2 decimal places. 1

Question 7 continued on next page.

Question 7 continued. **Marks.**

b) $P(x, y)$ is a point on the curve $y = e^{-x^2}$, where $x > 0$, O is the origin, and the perpendiculars from P to the *x*-axis and *y*-axis meet at A and B respectively.

(i) Show that the maximum area of the rectangle *OAPB is* $\frac{1}{\sqrt{2e}}$. 2

(ii) Show that the minimum length of *OP* is $\sqrt{\frac{1+\ln 2}{2}}$. 3

End of Question 7.

End of Examination.

Standard Integrals.

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x + \sqrt{x^2-a^2}\right),\ x > a > 0$$

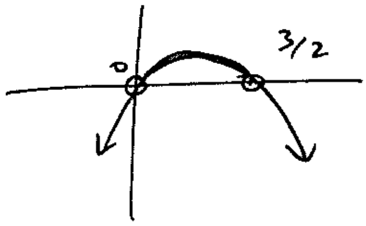
$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

SBHS THSC EXT I 2011

Question 1

(a) $x(3-2x) > 0$



$0 < x < \frac{3}{2}$

(b) $\frac{d}{dx}\left(e^{-x}\cdot\cos^{-1}x\right) = e^{-x}\cdot\frac{-1}{\sqrt{1-x^2}} + \cos^{-1}x\cdot -e^{-x}$

$= -e^{-x}\left(\cos^{-1}x + \frac{1}{\sqrt{1-x^2}}\right)$

(c) $P(x) = x^3 + ax^2 - 3x + 5$

$P(-2) = (-2)^3 + a(-2)^2 - 3(-2) + 5 = 11$

$-8 + 4a + 6 + 5 = 11$

$4a = 8$

$a = 2$

(d) $\int_0^{\frac{\pi}{8}} \sec 2x \tan 2x \, dx = \left[\frac{1}{2}\sec 2x\right]_0^{\frac{\pi}{8}}$

$= \frac{1}{2}\sec 2\left(\frac{\pi}{8}\right) - \frac{1}{2}\sec 2(0)$

$= \frac{1}{2}\sec\frac{\pi}{4} - \frac{1}{2}\sec 0$

$= \frac{1}{2}(\sqrt{2}) - \frac{1}{2}(1)$

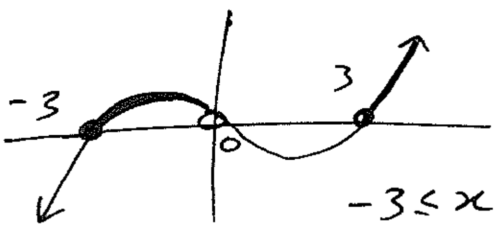
$= \frac{1}{2}(\sqrt{2} - 1)$

(e) $\frac{x^2-9}{x} \geq 0$

$x \neq 0$

$x(x^2-9) \geq 0$

$x(x-3)(x+3) \geq 0$



$-3 \leq x < 0, \ x \geq 3$

(f) $\int_0^2 \frac{1}{4+x^2}\,dx$

$= \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$

$= \frac{1}{2}\tan^{-1}\left(\frac{2}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{0}{2}\right)$

$= \frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{2}(0)$

$= \frac{\pi}{8}$.

2011 Mathematics Extension 1 Trial HSC Q2 soln

a)

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}.dx$$
$$= \int \frac{1}{\sqrt{1-u^2}}.du$$
$$= \sin^{-1} u + C$$
$$= \sin^{-1}(e^x) + C$$

$x = \ln u$
$e^x = u$
$du = e^x.dx$

b) $f(x) = x - \cos x$

$f'(x) = 1 + \sin x$

$$x_2 = \frac{1}{2} - \frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)}$$
$$x_2 = 0.5 + \frac{-0.37758}{1.479426}$$
$$x_2 = 0.76$$

c) $y = e^{2x}$

$y' = 2e^{2x}$

$y = 1 + 4x - x^2$

$y' = 4 - 2x$

At $(0, 1)$

$m_1 = 2, \qquad m_2 = 4$

$$\tan\theta = \left|\frac{2-4}{1+2(4)}\right|$$
$$= \left|-\frac{2}{9}\right|$$
$$\therefore \theta = 12°\,32'$$

d)

$$\frac{2}{\tan A + \cot A} = \sin 2A$$
$$LHS = \frac{2}{\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}}$$
$$= \frac{2}{\frac{\sin^2 A + \cos^2 A}{\sin A \cos A}}$$
$$= \frac{2}{\frac{1}{\sin A \cos A}}$$
$$= 2\sin A \cos A$$
$$= \sin 2A$$
$$= RHS$$

e)

$$\frac{d}{dx}[\cos^3 x°]$$

$$= \frac{d}{dx}\left[\cos^3\left(\frac{\pi x}{180}\right)\right]$$

$$= 3\cos^2\left(\frac{\pi x}{180}\right). -\frac{\pi}{180}\sin\left(\frac{\pi x}{180}\right)$$

$$= -\frac{\pi}{60}\cos^2\left(\frac{\pi x}{180}\right)\sin\left(\frac{\pi x}{180}\right)$$

QUESTION 3 $\times 1$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \cos(\alpha+\alpha) = \cos^2\alpha - \sin^2\alpha$

$= 1 - \sin^2\alpha - \sin^2\alpha$

$= 1 - 2\sin^2\alpha$

(ii) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

$= \lim_{x \to 0} - \frac{2\sin^2 x}{x^2}$

$= \lim_{x \to 0} = 2 \frac{\sin x}{x} \cdot \frac{\sin x}{x}$

$= 2$

(b.) $\tan(\alpha-\beta) = \frac{\frac{5}{12} - \frac{3}{4}}{1 + \frac{5}{12}\frac{3}{4}}$

$= \frac{-16}{3}$ or -0.254

(c) (i) $P_x = \frac{5k-1}{k+1}$ $P_y = \frac{-2k+7}{k+1}$

(ii) $\frac{25k-5}{k+1} = \frac{-8k+28}{k+1} - 1 = 0$

$33k - 33 = k+1$

$32k = 34$

$k = \frac{34}{32} = \frac{17}{16}$

$k = \frac{17}{16} : 1 = 17:16$

(d) if $n=1$

LHS $= 1 \times 1!$ RHS $= 2-1$

$S(1)$ is true.

Assume $S(k)$ true

$1\times1! + 2\times2! \ldots + k\times k! = (k+1)! - 1$

$S(k+1)$ is

$1\times1! + 2\times2! + k\times k! + (k+1)(k+1)! = (k+2)! - 1$

LHS $(k+1)! - 1 + (k+1)(k+1)!$ by assumption

$= (k+1)!\,[1 + k + 1] - 1$

$= (k+2)(k+1)! - 1$

$= (k+2)! - 1$ = RHS

$\therefore S(k+1)$ is true if $S(k)$ is true

$\therefore$ by Mathematical Induction, $S(n)$ is true for any integer $n \geq 1$

(e) $-1 \leq x \leq 1$

$1 \geq -x \geq -1$

$2 \geq 1-x \geq 0$

$0 \leq 1-x \leq 2.$

2011 Maths Ext 1 TRIAL — SOLNS

✓ $= \frac{1}{2}$ m

4 (a)

(i) $m_{PQ} = \dfrac{ap^2 - aq^2}{2ap - 2aq} = \dfrac{a(p^2 - q^2)}{2a(p-q)}$

$= \dfrac{(p-q)(p+q)}{2(p-q)}$

$\Rightarrow m_{PQ} = \frac{1}{2}(p+q)$ ✓

Then Equation of PQ: $y - y_1 = m(x - x_1)$

$y - ap^2 = \frac{1}{2}(p+q)(x - 2ap)$ ✓

$y - ap^2 = \frac{1}{2}(p+q)x - ap(p+q)$ ✓

$y - ap^2 = \frac{1}{2}(p+q)x - ap^2 - apq$

$\Rightarrow y = \frac{1}{2}(p+q)x - apq$

$\therefore y - \frac{1}{2}(p+q)x + apq = 0$ ✓

(2)

(ii) PQ passes through $(0, -a)$

$\Rightarrow -a - \frac{1}{2}(p+q)0 + apq = 0$ ✓

$-a + apq = 0$

$a(pq - 1) = 0$

$\Rightarrow a = 0$ or $pq = 1$

(1)

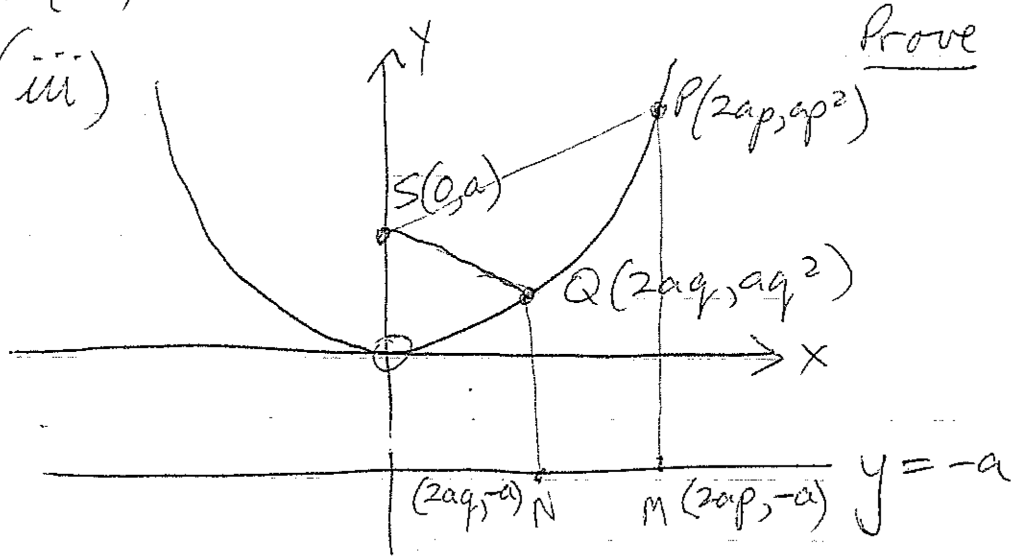
Not a parabola (for $a = 0$)

$\therefore \underline{pq = 1}$ ✓

4 (a)

(iii)

Prove $\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$.



From defn of a parabola,

$SP = PM$ and $SQ = QN$

Now $PM^2 = (ap^2+a)^2 + (2ap-2ap)^2$

$\Rightarrow PM = ap^2 + a$
$= a(p^2+1) = SP$.

Also, $QN = aq^2 + a$
$= a(q^2+1) = SQ$.

(2)

Then $\frac{1}{SP} + \frac{1}{SQ}$

$= \frac{1}{a(p^2+1)} + \frac{1}{a(q^2+1)}$

$= \frac{q^2+1+p^2+1}{a(p^2+1)(q^2+1)} = \frac{p^2+q^2+2}{a(p^2q^2+p^2+q^2+1)}$

But $pq = 1$ (from ii)

$\Rightarrow \frac{1}{SP} + \frac{1}{SQ} = \frac{p^2+q^2+2}{a(p^2+q^2+2)} = \frac{1}{a}$ #

4 (b) (i) [F | ... | M]

[M | ... | F]

2! ways parents are on ends.

Ways girls and boys are seated $= 4! \times 3!$

$\therefore$ Total ways of seating this way $= 2! \times 4! \times 3! = 288$ ways. ✓

Also, Total Ways without restrictions $= 9! = 362880$ ways.

$\therefore$ P(parents at ends, boys + girls seated alternately between) $= \frac{288}{362880}$

$= \frac{1}{1260}$ ✓

(1)

(ii)

$2 \times [4 \times 3] \times 1 \times [5!]$

(2 boys above 4×3; Seat mother under 2; Seat father under 1; Seat rest under $5!$)

$= 1440$ ways. $\times 2 = 2880$ ✓

(1)

No way without restriction $= (9-1)! = 8!$

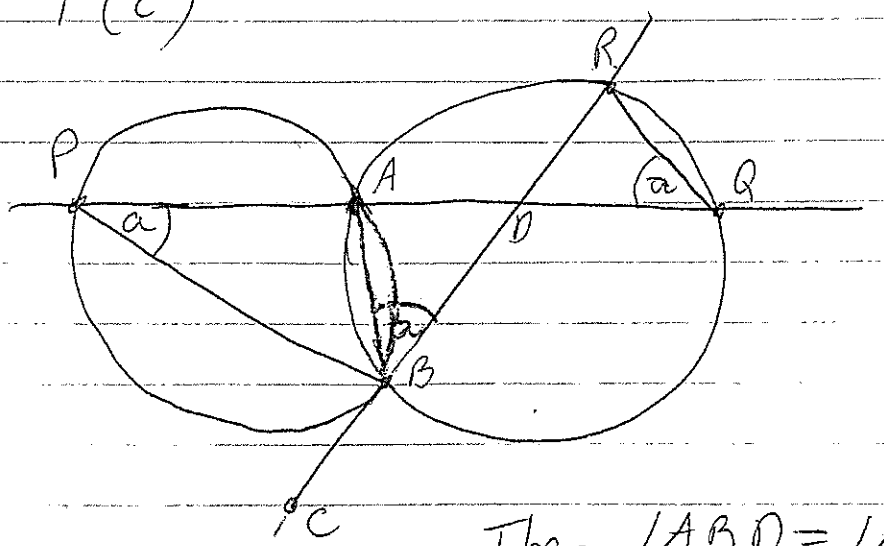
$= 40320$ way

$\therefore$ P(M and F separated by 2 boys in circle) $= \frac{2880}{40320} = \frac{1}{14}$ ✓

4 (c)

Prove PB // QR

Construct AB, BP, R



Then ∠ABD = ∠APB (Alternate ∠ Theore

Let these angles = a.

Also ∠RQA = ∠RBA (Angles at circumfere standing on same arc are equal)

But ∠RBA = ∠ABD (same angle)

= a

∴ ∠RQA = ∠ABD = a

∴ PB // QR (Alternate angles are equal

2

4(d) $x^3 + px^2 + qx + r = 0$

<u>given</u> α β γ Let roots be $\alpha, \beta, \alpha+\beta$

Then $2(\alpha+\beta) = -p$ (sum of roots $= \frac{-b}{a}$)

$\alpha+\beta = \frac{-p}{2}$.

But $(\alpha+\beta)$ is a root.

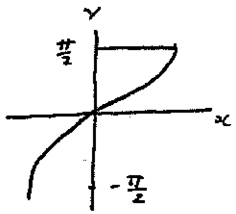
$\Rightarrow P\left(\frac{-p}{2}\right) = \frac{-p^3}{8} + \frac{p^3}{4} \frac{-pq}{2} + r = 0$

$\times 8 \Rightarrow -p^3 + 2p^3 - 4pq + 8r = 0.$

$\Rightarrow 4pq = p^3 + 8r$ #

(a) $y = \sin^{-1} x$

$\therefore x = \sin y$



$V = \pi \int_0^{\frac{\pi}{2}} x^2 . dy$

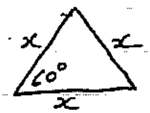
(i) $= \pi \int_0^{\frac{\pi}{2}} \sin^2 y . dy$

$= \pi \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos 2y) dy$

$= \frac{\pi}{2} \left[y - \frac{\sin 2y}{2} \right]_0^{\frac{\pi}{2}}$

(ii) $= \frac{\pi^2}{4}$

(b)



(i) $A = \frac{1}{2} . x^2 . \sin 60$

$= \frac{\sqrt{3} x^2}{4}$

(ii) $A = \frac{\sqrt{3} x^2}{4}$

$\frac{dA}{dx} = \frac{\sqrt{3} x}{2}$

$\frac{dA}{dt} = \frac{dA}{dx} . \frac{dx}{dt}$

$4 = \frac{\sqrt{3} x}{2} . \frac{dx}{dt}$

at $x = 2$; $\frac{dx}{dt} = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$

(c) $f(x) = x^2 - 6x + 13$

$f(x) = (x-3)^2 + 4$

Vertex at $(3,4)$

Largest domain of positive values $\{x : x \geq 3\}$

For inverse function $x = (y-3)^2 + 4$

$(y-3)^2 = x - 4$

$y - 3 = \sqrt{x-4}$

$y = 3 + \sqrt{x-4}$

$\therefore f^{-1}(x) = 3 + \sqrt{x-4}$

(d)(i) Show $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$

$\ddot{x} = \frac{dv}{dt}$

$= \frac{dv}{dx} . \frac{dx}{dt}$

$= \frac{dv}{dx} . v$

$= \frac{d(\frac{1}{2}v^2)}{dv} . \frac{dv}{dx}$

$= \frac{d(\frac{1}{2}v^2)}{dx}$

or $\frac{d(\frac{1}{2}v^2)}{dx} = \frac{d(\frac{1}{2}v^2)}{dv} . \frac{dv}{dt} . \frac{dt}{dx}$

$= v . \frac{dv}{dt} . \frac{1}{v}$

$= \frac{dv}{dt}$

$= \ddot{x}$

(ii) $\ddot{x} = 8x - 3x^2$

$\therefore \frac{d(\frac{1}{2}v^2)}{dx} = 8x - 3x^2$

By integration $\frac{1}{2}v^2 = 4x^2 - x^3 + c$

at $t=0, x=0, v=4$; $\therefore 8 = c$

$\therefore \frac{1}{2}v^2 = 4x^2 - x^3 + 8$

$v^2 = 8x^2 - 2x^3 + 16$

at $x = 1$; $v^2 = 8 - 2 + 16$

$v^2 = 22$

$v = \pm\sqrt{22}$

Speed (has no direction) ANS $= \sqrt{22}$

QUESTION 6 X1

(a) $y = 3\cos^{-1}(\sin 2x) - 2\sin^{-1}(\cos 3x)$

$y' = -3 \frac{\times 2\cos 2x}{\sqrt{1-\sin^2 2x}} - \frac{2\times -3\sin 3x}{\sqrt{1-\cos^2 3x}}$

$= -\frac{6\cos 2x}{\cos 2x} + \frac{6\sin 3x}{\sin 3x}$

$= 0$

$\therefore y$ constant

(b) $\tan 49 = \frac{h}{a}$ $\tan 41 = \frac{h}{b}$

$200^2 + \frac{h^2}{\tan^2 49} = \frac{h^2}{\tan 41}$

$h^2 \left(\frac{1}{\tan^2 41} - \frac{1}{\tan^2 49}\right) = 200^2$

$h^2 = 200^2 \div \left(\frac{1}{\tan^2 41} - \frac{1}{\tan^2 49}\right)$

$h \approx 265$

(c) (i) $x = a\cos(2t+\beta)$

$\dot{x} = -2a\sin(2t+\beta)$

$\ddot{x} = -4a\cos(2t+\beta)$

$= -4x$

(ii) $t=0$ $v=0$ $x=4$

$v^2 = n^2(a^2 - x^2)$

$4 = 4(a^2 - 16)$

$a = \sqrt{17}$

(iii) max velocity $= an$

$= 2\sqrt{17}$

(iv) $t = \tan\theta/2$

$\sin\theta = \frac{2t}{1+t^2}$ $\cos\theta = \frac{1-t^2}{1+t^2}$

$\sin\theta - \cos\theta = \frac{2t - 1 + t^2}{1+t^2} = 1$

$2t - 1 + t^2 = 1 + t^2$

$2t = 2$

$t = 1$

$\tan\frac{\theta}{2} = 1$

$\theta = \pi$

or

$\theta = (2n+1)\pi + \frac{\pi}{2}$

$n \in \mathbb{Z}$

(X1)

Q7. (i)

$\ddot{x} = 0$

$\dot{x} = c_1$

$\therefore \dot{x} = v\cos\alpha$ when $t=0$

$\therefore \dot{x} = v\cos\alpha$

$x = vt\cos\alpha + c_3$

$x=0, t=0$

$\therefore c_3 = 0$

$x = vt\cos\alpha$ ✓

$\ddot{y} = -10$

$\dot{y} = -10t + c_2$

$\dot{y} = v\sin\alpha$ when $t=0$

$\therefore c_2 = v\sin\alpha$

hence

$\dot{y} = -10t + v\sin\alpha$

$y = -5t^2 + vt\sin\alpha + c_4$

when $t=0, y=0$

$\therefore c_4 = 0$

hence $y = vt\sin\alpha - 5t^2$ ✓

When $t=0$

$\dot{x} = v\cos\alpha$

& $\dot{y} = v\sin\alpha$

$x = y = 0$.

(ii) when $t = 1.5$ $x = 6$ & $\dot{y} = 0$

$\therefore 6 = v \times \frac{3}{2}\cos\alpha$

$v\cos\alpha = 4$ ✓

& $0 = -10 \times \frac{3}{2} + v\sin\alpha$

$v\sin\alpha = 15$ ✓

(iii) $\sin^2\alpha + \cos^2\alpha = 1$

$\therefore \left(\frac{15}{v}\right)^2 + \left(\frac{4}{v}\right)^2 = 1$ ✓

$225 + 16 = v^2$

$v = \sqrt{241}$ m/s.

$\cos\alpha = \frac{4}{\sqrt{241}}$ ✓

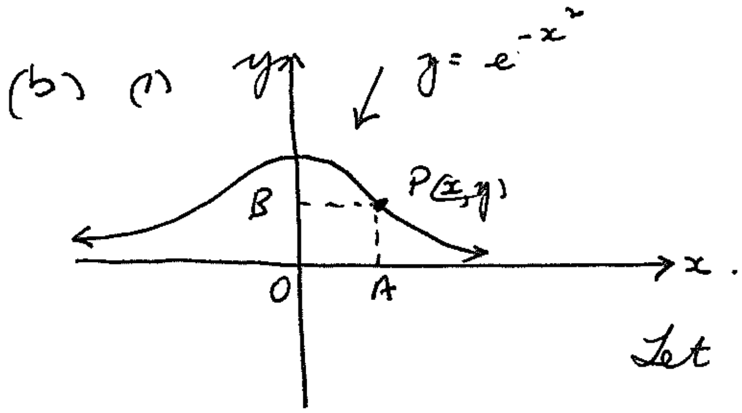
$\therefore \alpha = 75°$

$h = 2t\sin\alpha - 5t^2 + 1$

$= 2 \times 1.5 \times \ \ - 5 \times \frac{9}{4} + 1$

$= 12.25$ ✓

(b) (i)



Let A = area of OAPB

$= xy$

$= x e^{-x^2}$

$\frac{dA}{dx} = x \times -2x e^{-x^2} + e^{-x^2}$

$= e^{-x^2}(1 - 2x^2)$

Let $\frac{dA}{dx} = 0$

$1 - 2x^2 = 0$

$x^2 = \frac{1}{2}$

$x = \frac{1}{\sqrt{2}}$ (NB $x > 0$)

$\therefore A = \frac{1}{\sqrt{2}} e^{-\left(\frac{1}{\sqrt{2}}\right)^2}$

$= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{e}}$

$= \frac{1}{\sqrt{2e}}$ * [MUST TEST FOR MAX. USING NUMBERS]

(ii) $l_{OP} = \sqrt{x^2 + y^2}$

$= \sqrt{x^2 + (e^{-x^2})^2}$

$= \left(x^2 + e^{-2x^2}\right)^{\frac{1}{2}}$

$l' = \frac{1}{2}\left(x^2 + e^{-2x^2}\right)^{-\frac{1}{2}} \times \left(2x - 4x e^{-2x^2}\right)$

$$= \frac{2x\left(1-2e^{-2x^2}\right)}{2\sqrt{x^2+e^{-2x^2}}}$$

let $l' = 0$

ie. $2x\left(1-2e^{-2x^2}\right) = 0$

$x = 0$ or $e^{-2x^2} = \frac{1}{2}$.

↓

then $OP = 1$.

$$e^{-2x^2} = \frac{1}{2}.$$

$$e^{2x^2} = 2.$$

$$2x^2 = \ln 2.$$

$$x^2 = \frac{\ln 2}{2}$$

$$x = \sqrt{\frac{\ln 2}{2}}, \quad (x > 0)$$

$$\therefore l = \sqrt{\frac{\ln 2}{2} + e^{-2\frac{\ln 2}{2}}}$$

$$= \sqrt{\frac{\ln 2}{2} + e^{-\ln 2}}$$

$$= \sqrt{\frac{\ln 2}{2} + e^{\ln \frac{1}{2}}}$$

$$= \sqrt{\frac{\ln 2}{2} + \frac{1}{2}}$$

$$= \sqrt{\frac{1+\ln 2}{2}}$$

(NB MUST TEST FOR MIN USING VALUES)